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Metrizability of spaces having certain k -networks

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As is well-known, each of the following properties implies that X is metrizable.

- (A) X is a paracompact developable space (R. H. Bing [2]).
- (B) X is a paracompact space having a σ -locally countable base (V. V. Fedorcuk [9]).
- (C) X has a σ -hereditarily closure-preserving base (D. Burke, R. Engelking and D. Lutzer [5]).
- (D) X is a paracompact M -space, and a σ -space (A. Okuyama [15]; the paracompactness can be omitted; see F. Siwiec and J. Nagata [16]).
- (E) X is an M -space having a point-countable base (V. V. Filippov [10]).

In terms of these properties, we give some metrization theorems by means of certain k -networks, or generalizations of M -spaces, etc.

We assume that spaces are *regular and T_1* .

Definitions. (1) A cover \mathcal{C} of a space a *k -network* if, whenever $K \subset U$ with K compact and U open in X , then $K \subset \bigcup \mathcal{C}' \subset U$ for some finite $\mathcal{C}' \subset \mathcal{C}$. If the K is a single point, then such a cover is called network (or net). Recall that a space is an \mathfrak{K} -space (resp. σ -space) if it has a σ -locally finite k -network (resp. network).

(2) A space is *countably bi-quasi-k* [14] if, whenever $\{F_n: n \in \mathbb{N}\}$ is a decreasing sequence with $x \in \overline{F_n}$, there exists a decreasing sequence $\{A_n: n \in \mathbb{N}\}$ such that $x \in \overline{A_n \cap F_n}$ for each $n \in \mathbb{N}$, and if $x_n \in A_n$ for each $n \in \mathbb{N}$, then the sequence $\{x_n: n \in \mathbb{N}\}$ has a cluster point in $\bigcap \{A_n: n \in \mathbb{N}\}$. Recall that a space X is a q -space if each point has a sequence $\{V_n: n \in \mathbb{N}\}$ of nbds such that if $x_n \in V_n$, the sequence $\{x_n: n \in \mathbb{N}\}$ has a cluster point in X . Every q -space is a countably bi-quasi-k-space. Every countably bi-quasi-k-space is precisely the countably bi-quotient image of an M -space; see [14].

(3) A space X is a *monotonic $w\Delta$ -space* (simply, $mw\Delta$ -space) [18] (resp. *monotonic developable* [7] (equivalently, space having a base of countable order in the sense of Arhangel'skii [1])), if there exists a sequence (\mathcal{B}_n) of bases for X such that any *decreasing* sequence $\{B_n: n \in \mathbb{N}\}$ with $B_n \in \mathcal{B}_n$ satisfies (*):

(*) If $x \in \bigcap B_n$, and $x_n \in B_n$, then the sequence $\{x_n: n \in \mathbb{N}\}$ has a cluster point in X (resp. the cluster point x).

If the sequence $\{B_n: n \in \mathbb{N}\}$ with $B_n \in \mathcal{B}_n$ is not necessarily decreasing, then such a space is called a $w\Delta$ -space; developable space respectively.

Every $w\Delta$ -space or every (monotonic) p -space [7] is an $mw\Delta$ -space. Every $mw\Delta$ -space is a q -space, hence countably bi-quasi-k. Among θ -refinable (= submetacompact) spaces, $mw\Delta$ -spaces (resp. monotonic developable spaces) are $w\Delta$ -spaces (resp. developable spaces); see [18] (resp. [7]).

The lemma below holds by means of [13; Proposition 3.2], [17; Section 4], [4; Theorem 4.1], and [18], etc. Here, we note that, in a space having a σ -locally countable k -network, each point is a G_δ -set. So, every countably bi-quasi-k-space with a σ -locally countable k -network is countably bi-k by [14; Theorem 7.3].

Lemma. (1) Suppose that X is a k -space; a normal space in which every closed countably compact set is compact; or each point of X is a G_δ -set. Then (i) and (ii) below hold.

(i) X has a point-countable base if and only if X is a countably bi-quasi- k -space with a point-countable k -network.

(ii) X is a monotonically developable space with a point-countable base if and only if X is an $mw\Delta$ -space with a point-countable k -network (cf. [18]).

(2) (i) A space has a σ -locally countable base if and only if it is a countably bi-quasi- k -space with a σ -locally countable k -network.

(ii) A space is a monotonically developable space with a σ -locally countable base if and only if it is an $mw\Delta$ -space with a σ -locally countable k -network.

Metrizability Theorem The following are equivalent.

- (a) X is metrizable,
- (b) X is a paracompact M -space having a point-countable k -network,
- (c) X is an M -space, and a k -space having a point-countable k -network,
- (d) X is an M -space having a point-countable k -network, and having a σ -locally countable *network*.
- (e) X is an M -space having a σ -locally countable k -network,
- (f) X is a paracompact, countably bi-quasi- k -space having a σ -locally countable k -network,
- (g) X is a countably bi-quasi- k -space having a point-countable k -network, and having a σ -closure preserving k -network; cf. [12].
- (h) X is an $mw\Delta$ -space having a σ -closure preserving k -network; see [18].
- (i) X is a countably bi-quasi- k -space having a σ -hereditarily closure preserving k -network.

Remark. In the previous theorem, it is possible to replace “ countably bi-quasi-k-space ” by “ countably bi-quasi-k-space, but the sequence $\{x_n: n \in \mathbb{N}\}$ has a cluster point in X (instead of $\bigcap \{A_n: n \in \mathbb{N}\}$) in the definition of countably bi-quasi-k-spaces ”.

We see that each condition in (b) \sim (h) of Metrization Theorem is essential by means of the following examples.

Examples. (1) Not every countably compact (resp. countably compact, first countable) space having a point-countable k-network (resp. locally countable *network*) is a σ -space.

(2) Not every Čech-complete (resp. metacompact developable) space having a σ -locally countable base is a σ -space [8] (resp. metrizable; cf. [11]).

(3) Not every first countable, Lindelöf space having a σ -closure preserving base is developable [6].

But, the following holds by means of [3] and [18], etc.

Proposition. (1) Every θ -refinable space X is developable if (a) or (b) below holds.

(a) X is an $\text{mw}\Delta$ -space having a point-countable k-network, or having a σ -locally countable *network*.

(b) X is a countably bi-quasi-k-space having a σ -locally countable k-network.

(2) Every $\text{mw}\Delta$ -space which is the quotient compact image (resp. quotient s-image) of a metric space is developable (resp. monotonically developable); see [18].

We note that every $mw\Delta$ -space having a σ -locally finite *network* (resp. σ -locally countable k -network) is developable (resp. monotonically developable; cf. [18]).

In view of the above results, we have the following questions.

Questions. (1) Every $w\Delta$ -space having a σ -locally countable k -network (resp. σ -locally countable *network*) is developable (resp. monotonically developable) ?

(2) Every $w\Delta$ -space which is the quotient countable-to-one image of a metric space is developable ?

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